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MEMBRANE DEFORMATION BY A MOVING LOAD†

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Small steady deformations (displacements) of the elements of a homogeneous membrane caused by a [force] load moving uniformly along one of the axes with velocity exceeding the velocity of propagation of elastic waves in the membrane are investigated. The cases of a load distributed along the other axis and concentrated loads are considered. Deformations for an unbounded membrane, a half-plane and an unbounded strip are analysed. A method is used which enables a new independent variable to be introduced and enables a problem equivalent to the action of a moving load or a system of moving loads on an unbounded or half-bounded string to be obtained. Problems with central symmetry are also considered. Namely, it is assumed that an undeformable disc is attached rigidly to an unbounded membrane and a concentrated [force] load is moving at constant velocity around a circle with the same centre as the disk. An investigation of the problem in polar coordinates enables it to be reduced to the general form of the problem of the deformation of a string. By applying the appropriate results obtained earlier for a string [1] the following qualitative result is established: there are values of the velocity of motion of the load such that no work is performed in overcoming wave resistance forces in the steady state, which is equivalent to the absence of such forces.

Small displacements $u(x, y, t)$ of a membrane perpendicular to its plane (x, y) satisfy the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \quad a^2 = \frac{T}{\rho} \tag{1}$$

where T is the tension of the membrane and p is its surface density.

Suppose that a load distributed with constant density f with respect to x in an interval $|x| \leqslant \delta$ moves along the y axis with constant velocity υ greater than the velocity of propagation of a wave a in the membrane. Under this load

$$T\frac{\partial u}{\partial y}\Bigg|_{y=\upsilon t-0}^{y=\upsilon t+0} = \begin{cases} -f, & |x| \leqslant \delta \\ 0, & |x| \geqslant \delta \end{cases} \tag{2}$$

where the minus sign corresponds to pressing the membrane from above.

We will be concerned with the steady state of membrane deformations. Thus, it will be convenient to consider problem (1), (2) in a new variables x, $\xi = y - \upsilon t$, that is, in a moving system of coordinates, where the membrane deformation pattern is independent of time. After these transformations problem (1), (2) takes the form

$$\frac{\partial^2 u}{\partial \xi^2} = b^2\frac{\partial^2 u}{\partial x^2}, \quad b^2 = \frac{a^2}{\upsilon^2 - a^2} \tag{3}$$

$$\frac{\partial u}{\partial \xi}\Bigg|_{\xi=-0}^{\xi=+0} = \psi(x, 0) = \begin{cases} -\varphi_0 = -f/T, & |x| \leqslant \delta \\ 0, & |x| > \delta \end{cases}$$

In this formulation the problem of membrane deformations is identical with the well-known problem of the vibrations of an infinite string (e.g. [2]) without initial deformation, whose points $|x| \leqslant \delta$ move with the same initial velocity. By a formal substitution of ξ for at in the problem for a string we arrive at problem (3), so we can use the known solution.

In a system of coordinates with the moving load the deflection of the membrane is given by

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$$u(x,\xi)=\frac{1}{2b}\int_{x-b\xi}^{x+b\xi}\psi(\alpha)d\alpha$$

It follows that $u(x, \xi)$ is given in the form of two "waves"

$$u(x,\xi)=\Phi(x+b\xi)-\Phi(x-b\xi)$$

$$\Phi(\beta)=\begin{cases}0, & \beta<-\delta\\ -\varphi_0(\beta+\delta)/(2b), & |\beta|\leq\delta\\ -\varphi_0\delta/b, & \beta>\delta\end{cases}$$

In the (x, ξ) plane we take the characteristics passing through the end points of the interval AB (Fig. 1). In the problem in question they have the form $x + b\xi = C_1$, $x - b\xi = C_2$, where C_1 and C_2 are constants. The characteristics shown in Fig. 1 divide the (x, ξ) plane into domains 1 and 2, in which there are no deformations, and domains 3–6, where deformations exist. In domain 6 the deformations are the same for all points, while in 3–5 they are linear in x. In Fig. 1 we also show the profiles of the membrane for various ξ = const.

Relative to the fixed system of coordinates (connected with the membrane) we can observe that, as time elapses, large domains of the membrane become deformed during the motion of the load, and work is performed by the load. This can be explained by the fact that the force distributed over the interval AB is inclined in the opposite direction to the motion, i.e. it has a horizontal component f_1. It follows that there is a force of equal magnitude resisting the motion of the load.

To compute f_1 we turn to Fig. 1. The forces acting in the interval AB are

$$\mathbf{T}_1=T\mathbf{e}_\xi,\quad \mathbf{T}_2=-T(\cos\alpha\mathbf{e}_\xi+\sin\alpha\mathbf{e}_u),\quad \mathbf{f}=f_1\mathbf{e}_\xi-f_2\mathbf{e}_u$$

per unit length. Here $\mathbf{e}_\xi$ and $\mathbf{e}_u$ are unit vectors. In a time interval dt the forces give rise to the following variation of the momentum of an element of the membrane of length $d\xi = \upsilon\, dt$

$$(\mathbf{T}_1+\mathbf{T}_2+\mathbf{f})dt=\rho(\mathbf{v}_2-\mathbf{v}_1)d\xi$$

$$\mathbf{v}_1=-\mathrm{v}e_\xi,\quad \mathbf{v}_2=-\upsilon(\cos\alpha\mathbf{e}_\xi+\sin\alpha\mathbf{e}_u)$$

Hence we find that $f_1 = T(1 - \cos\alpha)/b^2$, $f_2 = T\sin\alpha/b^2$. For $\alpha \ll 1$ we have $f_1 = T\alpha^2 2b^2 = f^2b^2/(2T)$.

The resistive force can also be computed from the energy flux through the line x = const and the given velocity. As the length of AB decreases, the load approaching a concentrated one, and it

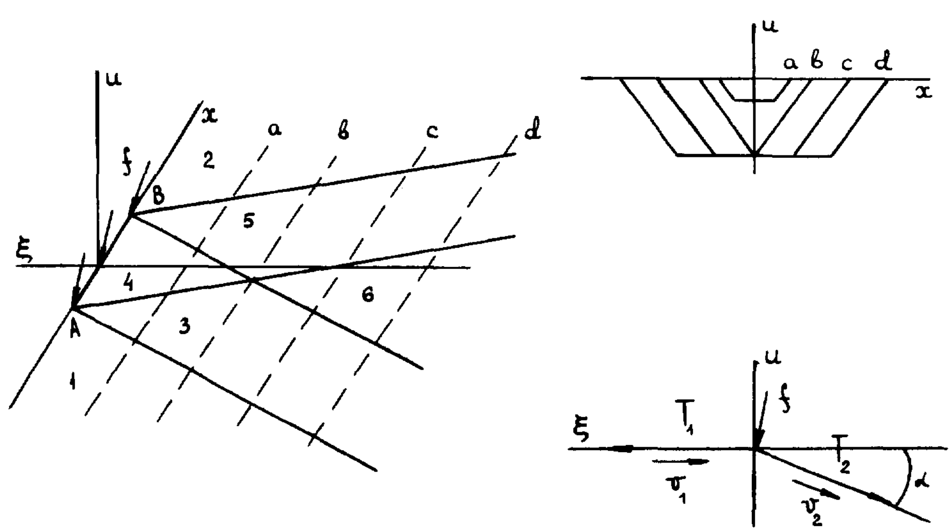

Fig. 1.

becomes impossible to determine the position of f, and hence the component f_1, by the above method. In this case f_1 must be computed from the energy flux.

In the special case of a point load, which will be considered below, $\Phi(\beta)$ has the form of the unit function and the membrane deformation pattern simplifies (Fig. 2). Inside the hatched area the membrane is deflected, the displacement being the same at each point. Outside this area there is no deflection.

When the membrane rests of an elastic support, its deformation will no longer be so simple, but the boundary of the perturbed domain (the shape of the characteristics) remains unchanged.

The results of a study of wave propagation in a half-bounded string [1] can also be applied to the problem of membrane deformation. Suppose that the membrane is bounded by the straight line $x = -x_0(x_0 > 0)$, which is represented by a line with shading. In this case problem (3) must be supplemented by the condition that the membrane is not subject to any displacements on the boundary: $u = (-x_0, y, t) = 0$. The mathematical method which enables the problem to be solved, involves the change from a bounded membrane to an unbounded one in which the condition under the load (3) is extended symmetrically about the straight line $x = -x_0$ but not evenly to the whole straight line $\xi = 0$.

In the case of a point load the deformations of a bounded membrane will be the same as those of an unbounded membrane in the domain $x \geqslant -x_0$ subject to a load given at the point $x = 0$, $\xi = 0$ and a load of opposite sign at the point $x = -2x_0$, $\xi = 0$ symmetrical about $x = -x_0$. The membrane deformation pattern is shown in Fig. 3. In the hatched domain the membrane is subject to negative deflection of constant magnitude and remains unperturbed in any other region in the half-plane.

When the load system moves along the membrane boundary $x = -x_0$, the deformation pattern can be obtained as the superposition of the deformations of each of the loads, having the form shown in Fig. 3. In the case when there is an infinite system of identical loads distributed with period $2x_0/b$ (Fig. 4, in which the loads are shown by circles with crosses inside), the deformations are the same at every point of the membrane for $x > 0$. Relative to the fixed system of coordinates (x, y) the membrane

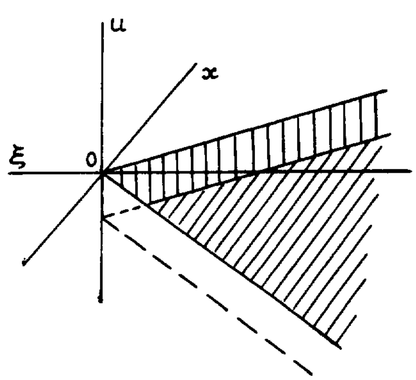

Fig. 2.

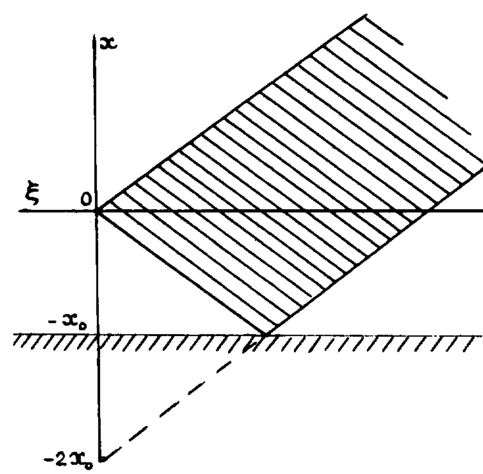

Fig. 3.

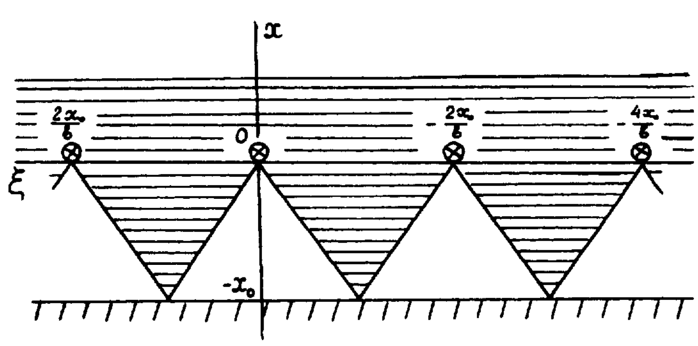

Fig. 4.

deformation pattern will also be invariant for $x > 0$. Therefore there is no energy flux through any line $x = \text{const} > 0$, i.e. no work is performed by the moving system of loads. In other words, in this case the system of loads moving parallel to the "wall" $x = 0$, $\xi = 0$ does not experience any resistance.

Finally, we consider a strip-shaped membrane $|x| \leq x_0$ subject to a moving point load at the same distance to the boundaries. This problem can also be reduced to the problem of the deformations of an unbounded membrane acted upon by a system of loads along the straight line $\xi = 0$. A load is applied at $x = 0$, and the remaining "loads" appear as a result of the uneven reflection of the conditions at the point $x = 0$ about the boundaries of the membrane and subsequent second-order uneven reflections, third-order reflections, and so on. This leads to the deformation pattern of the strip shown in Fig. 5. The membrane is divided into rhomboidal domains. Outside these domains there are no deformations, while inside them u is constant and the deflection sign alternates.

Now suppose that two identical loads separated by a distance $2x_0/b$ move with constant velocity along the central line of the membrane. Each of these loads separately gives rise to the deformations shown in Fig. 5. The superposition principle holds due to the linearity of the system. Under the joint action of the two loads, only the rhomboidal domain between the loads will be perturbed (Fig. 6). Outside this domain the deflection u vanishes. By the physical arguments presented above we conclude that no work is performed jointly by the loads placed in this way as they move along the strip without meeting any resistance. A similar situation occurs for two loads separated by a distance equal to an odd multiple of the diagonal of the rhombus, $(2n + 1)2x_0/b$, except that there will be $2n + 1$ rhomboidal domains between the loads in which the membrane is subject to perturbation.

The deformation of the strip shown in Fig. 6 can also be achieved for an unbounded membrane using a system of four identical loads of different directions. The rhombus is formed by the characteristics corresponding to the loads at its vertices (see Fig. 7, where the directions of the loads are indicated by circles with dots and crosses inside).

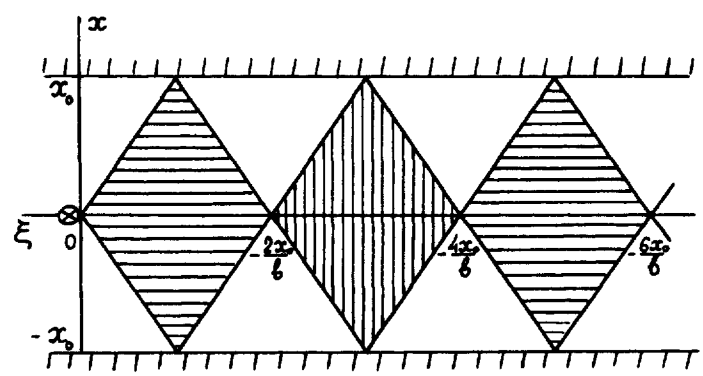

Fig. 5.

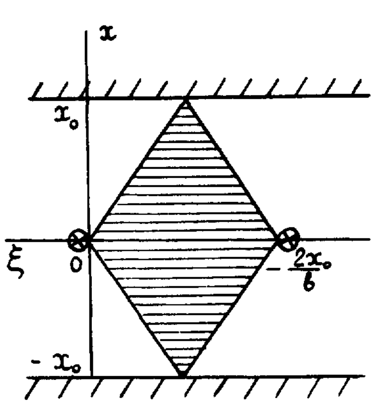

Fig. 6.

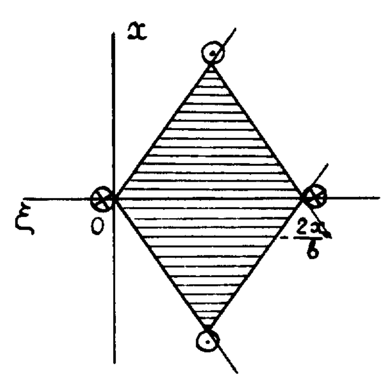

Fig. 7.

Now we consider a membrane of unbounded radius attached to an undeformable disc of radius r_0. A point load moves at constant velocity υ around a circle concentric with the disc on the membrane at a distance R_* from its centre.

The equation for the deflection u of the membrane has the form

$$\frac{\partial^2 u}{\partial t^2}=a^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2}\right) \tag{4}$$

in polar coordinates r, φ. Since we are dealing with steady-state membrane deformations, it will be convenient to consider the problem in a system of coordinates rotating about the axis through the centre of the disc, in which the load is stationary ($\omega = \upsilon/R_*$ being the angular velocity of the system) and the deflection is $u = u(r, \theta = \varphi - \omega t)$.

Equation (4) becomes

$$r^2\frac{\partial^2 u}{\partial r^2}+r\frac{\partial u}{\partial r}+(1-k^2r^2)\frac{\partial^2 u}{\partial \theta^2}=0 \tag{5}$$

Here we have changed to the length scale r_0, i.e. $1 \leqslant r < \infty$, $k = \omega r_0/a$, retaining the symbols for the deflection and the variable r.

At the point $r = r_* = R_*/r_0$, $\theta = 0$ where the load is applied

$$u_\theta\Big|_{\theta=-0}^{\theta=+0}=\begin{cases}0, & |x|>0\\ -\varphi_0, & x=0\end{cases} \tag{6}$$

The solution must satisfy the condition $u(1, \theta) = 0$ on the edge of the disc.

We assume that $k \geqslant 1$, i.e. Eq. (5) is hyperbolic for each point of the membrane. To solve problem (5), (6) we apply the method of characteristics (see, for example, [3]). Unlike the problems considered before, in which the loads moved along straight lines (characteristics), here the two families of characteristics are curvilinear and have the form

$$\eta_1=-\theta+\zeta(kr)=\text{const},\quad \eta_2=\theta+\zeta(kr)=\text{const}$$
$$(\zeta(x)=\sqrt{x^2-1}-\arccos(1/x))$$

In (5) we change to the variables η_1 and η_2. Starting from the solution of problem (3), we find that the function

$$u(r,\theta)=\Phi(\eta_1)-\Phi(\eta_2),\quad \Phi(\zeta)=\begin{cases}0, & \zeta<0\\ -\varphi_0, & \zeta>0\end{cases}$$

satisfies this equation outside the boundary of the perturbed domain.

In the idealized case of a point load the membrane deformation has a discontinuity at the boundary of the domain. This flaw can be removed by considering a load distributed over a finite interval, which ensures a continuous passage between the domains of the membrane with different kinds of deformation, as in the case of a load moving along a straight line.

The domain where the membrane is perturbed is bounded by two characteristics starting from (r_*, 0) which have the form of an untwisting curve (η_1 = const) and a twisting curve (η_2 = const) up to

$$\theta=\theta_0=\zeta(kr_*)-\zeta(k)$$

when the characteristic (η_2 = const) reaches the disc $r = 1$ (Fig. 8).

The reflection phenomenon must be used to obtain the complete pattern of membrane deformations. For $\theta > \theta_0$ the expression for the characteristic has the form

$$-\theta+\zeta(kr)=-2\theta_0+\zeta(kr_*)$$

For $\theta_0 < \pi$ the membrane deformation is shown in Fig. 8. It is negative and has the same magnitude

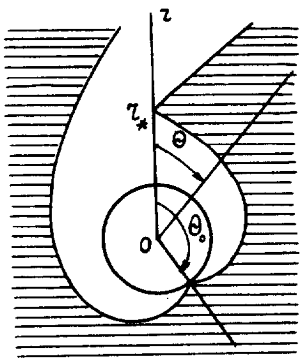

Fig. 8.

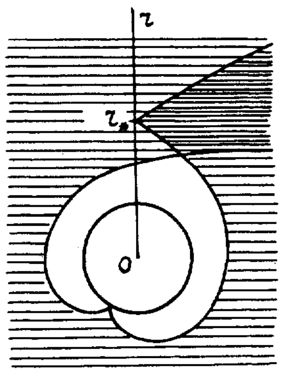

Fig. 9.

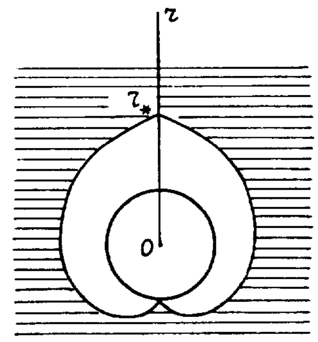

Fig. 10.

at each point of the hatched domain. For $\pi < \theta_0 < 2\pi$ the deformation pattern is shown in Fig. 9. In the domain adjoining the disc there is no deformation. In other domains it is negative. In the sector including the point where the load is applied the deformation is twice as large as outside this sector (the dense hatching in Fig. 9). Apart from those considered above, other situations are possible when the characteristic (η_2 = const) starting from $(r_*, 0)$ reaches the disc after one, two, or more revolutions about it ($\theta_0 > 2\pi$). A common feature in all cases (including those in Figs 8 and 9) is the presence of domains in θ with different forms of deformation for any fixed r.

The case of reflection at the point $\theta_0 = \pi$, when the reflected characteristic and the characteristic $\eta_1 = \zeta(kr_*)$ starting from $(r_*, 0)$ coincide, deserves to be considered separately. This leads to the membrane deformation pattern shown in Fig. 10. For $r > r_*$ the membrane deflection is constant at each point for any θ. Physically this means that there is no energy flux through any curve surrounding the unperturbed domain, that is, no work is performed by the load.

A similar situation occurs for $\theta_0 = 3\pi, \ldots, (2n + 1)\pi, \ldots$ also. This corresponds to one, two, etc. revolutions of the characteristic about the disc before it reaches the circle $r = 1$. From the condition $\theta_0 = (2n + 1)\pi$ for the estimates of υ_n we find that

$$\upsilon_n \simeq a\pi(2n+1)/(r_* - 1)$$

It follows that for the given position r_* of the load a discrete spectrum of velocities exists, for which no work is performed by the moving load, i.e. the motion does not meet any resistance. Another interpretation of the expression for υ_n is that for a given ω there is a denumerable set of circles $r_{*n} = 2\pi na/\omega r_0$ such that no work is performed by a load moving around them. Here we would like to point out the analogy with the steady electron orbits in the quantum theory of atomic structure, which also do not involve any energy radiation during the motion.

The effect obtained above is a result of the interaction between the perturbation generated by the load and the perturbation reflected from the boundary. For this to happen, at least two loads are necessary in the case of rectilinear motion, while one load is sufficient in the case of circular motion.

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